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Correlations in multifractals

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Abstract. We present a detailed analysis of the implicit assumptions that lie at the basis of the proposed scaling relations for multifractal correlations. Our results show the validity of this scaling relation but also the presence of non-trivial amplitude properties. The conclusions of the present work are valid for general multifractal systems and may have particular relevance for the field of turbulence where these correlations are experimentally available.

1. Introduction

The properties of spatial correlation functions of multifractals or fractal measures have been recently considered by Cates and Deutsch (1987). By using rather intuitive arguments they were able to derive certain scaling relations for these correlation functions. Similar relations had also been derived for the scaling properties of correlations in Hamiltonian systems by Wegner (1985).

Here we will present a more rigorous analysis of these correlations for multifractals. First the assumptions that lead to the scaling behaviour (Cates and Deutsch 1987) are formulated in a mathematical framework. We then analyse their validity in some particular but representative examples. The results are that: (i) the scaling properties obtained by Cates and Deutsch are indeed correct and (ii) the amplitude of the correlations can also be calculated exactly and it depends explicitly on the moments considered in the correlation functions.

The paper is structured as follows. In § 2 we present a general discussion of spatial correlations in multifractals. In § 3 some exact relations concerning these correlations are discussed. In § 4 we present an exact calculation for a specific example. Section 5 contains the summary.

2. General discussion

In this section we give an explicit mathematical formulation to the assumption needed to derive the scaling behaviour of Cates and Deutsch (1987). In the following section we are going to discuss in detail these assumptions for the case of the generalised Cantor set (GCS).

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Consider a fractal measure (distribution) embedded in a one-dimensional space with an overall size R and a lower cutoff a. We are interested in the properties of spatial correlation functions of the type

$$C_{mn}(r) = \langle \mu(x)^m \mu(x+r)^n \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} \mu(x_i)^m \mu(x_i+r)^n \tag{2.1}$$

where $\mu(x_i)$ is the total measure in a box of size a around a point $x_i = ia$, $N = R/a$ and r is an integer multiple of a. Note that we use the unrestricted correlation function. This means that our sum runs over both empty and non-empty boxes. By subdividing the system in boxes of size r it is useful to introduce the following notations. We define $\tilde{\mu}(r_k)$ as the total measure within the kth box of size r. By $\mu_r(x_i^k)$ we indicate the measure of the ith box of size a within the kth box of size r where $k = 1, 2, \ldots, Na/r$. As a result we have $N_r = R/r$ boxes of size r, each containing $N_a = r/a$ boxes of size a. This implies

$$\tilde{\mu}(r_k) \equiv \sum_{i=1}^{r/a} \mu_r(x_i^k) \equiv \sum_{i=(k-1)r/a}^{kr/a} \mu(x_i). \tag{2.2}$$

We can now rewrite (2.1) in the following form:

$$C_{mn}(r) = \frac{1}{N_r} \sum_{k=1}^{N_r} \left(\frac{1}{N_a} \sum_{i=1}^{N_a} \mu_r(x_i^k)^m \mu_r(x_i^{k+1})^n \right) = \langle\langle \langle \mu_r(x_i^k)^m \mu_r(x_i^{k+1})^n \rangle \rangle\rangle \tag{2.3}$$

where the double chevrons denote averages over boxes of size r while the single chevrons refer to averages over boxes of size a. Since the two measures in (2.3) refer to different boxes of size r one might assume that they are uncorrelated, i.e. (assumption I)

$$C_{mn}(r) = \langle\langle \langle \mu_r(x_i^k)^m \rangle \langle \mu_r(x_i^{k+1})^n \rangle \rangle\rangle. \tag{2.4}$$

We now multiply and divide (2.4) by the same quantity

$$C_{mn}(r) = \left\langle\left\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \left\langle \left(\frac{\mu_r(x_i^k)}{\tilde{\mu}(r_k)} \right)^m \right\rangle \left\langle \left(\frac{\mu_r(x_i^{k+1})}{\tilde{\mu}(r_{k+1})} \right)^n \right\rangle \right\rangle\right\rangle. \tag{2.5}$$

Now the terms within single chevrons have all been normalised and in view of the assumed self-similarity of the distribution these terms are now all equal and independent of k. This allows us to bring them outside the double chevons and to eliminate the index k within the single chevrons. We thus obtain

$$C_{mn}(r) = \langle\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \rangle\rangle \left\langle \left(\frac{\mu_r(x_i)}{\tilde{\mu}(r)} \right)^m \right\rangle \left\langle \left(\frac{\mu_r(x_i)}{\tilde{\mu}(r)} \right)^n \right\rangle. \tag{2.6}$$

The second assumption is to assume that we can replace the index $k+1$ by k in the double chevrons, i.e. (assumption II)

$$\langle\langle \tilde{\mu}(r_k)^m \mu(r_{k+1})^n \rangle\rangle \approx \langle\langle \tilde{\mu}(r_k)^{m+n} \rangle\rangle. \tag{2.7}$$

As we are going to see later this assumption is only valid for the scaling behaviour while the amplitude will be shown to depend explicitly on m and n. Using (2.7) together with the standard scaling relations for the moments of multifractals (Halsey *et al* 1986)

$$\langle\langle \tilde{\mu}(r_k)^m \rangle\rangle \sim (R/r)^{-\tau(m)-1} \tag{2.8}$$

$$\left\langle \left(\frac{\mu_r(x_i)}{\tilde{\mu}(r)} \right)^m \right\rangle \sim (r/a)^{-\tau(m)-1} \tag{2.9}$$

one readily obtains the final result

$$C_{mn}(r) \sim (R/a)^y (r/a)^z \tag{2.10}$$

$$y = -\tau(m+n) - 1 \tag{2.11}$$

$$z = \tau(m+n) - \tau(m) - \tau(n) - 1. \tag{2.12}$$

Note that if one had used the restricted definition for (2.1), implying summing only over non-zero measures, then (2.11) and (2.12) would show D_0 instead of 1, where D_0 is the fractal dimension of the set on which the measure is not zero.

3. Exact properties of $C_{mn}(r)$ for the generalised Cantor set

In order to get some insight into the properties of the correlation function (2.1) it is useful to consider the specific example of the generalised Cantor set (Halsey *et al* 1986, Siebesma and Pietronero 1986, Pietronero and Siebesma 1986). This archetype of a multifractal can be considered as the generalisation of the Cantor set in the sense that different weights are given to the different segments in a multiplicative way. From now we are going to consider for simplicity only the specific case shown in figure 1

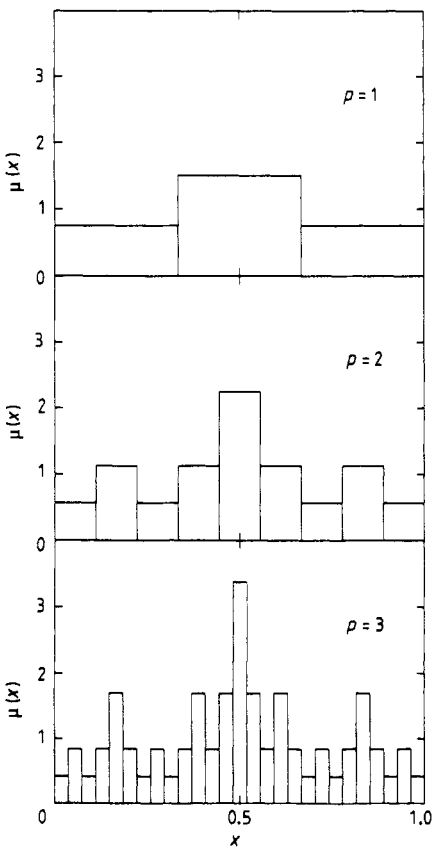

Figure 1. First steps of the construction of the generalised Cantor set used as an example for the discussion of multifractal correlations.

where at every stage a segment is divided in three new segments of length one-third of the previous one, the central one having twice the weight of each of the lateral ones. It should be noted, however, that all the results we are going to obtain for this particular example are easily extendable to many more cases including GCS in which the dimension of the support of the measure is less than one.

The overall size is $R=1$ and the lower cutoff is $a=(\frac{1}{3})^p$, where p is the order of the iteration. In figure 2(a) the function $C_{mn}(r)$ (equation (2.1)) is shown for $p=9$ and $m=n=2$. At first sight $C_{mn}(r)$ appears rather irregular but one can see from its integral (figure 2(b)) that a well defined power law can be associated with it in the average sense. Concerning the detailed structure of $C_{mn}(r)$ one can easily observe that $C_{mn}(r)=$ constant for discrete sets of values of type

$$r_j \in (x_j; x_{3j}; x_{3j^2}; \ldots) \qquad j=1,2,\ldots. \tag{3.1}$$

Such a property can be easily proven by using some permutation symmetries. Consider therefore the GCS as a string of values where the ith value corresponds to the measure in the ith box. We then introduce the string X defined as the set of N values

$$X \equiv (x_1; x_2; \ldots; x_N) \tag{3.2}$$

and the corresponding multiplication rules

$$x_0 X = (x_0 x_1; x_0 x_2; \ldots; x_0 x_N) \qquad YX \equiv \sum_{i=1}^{N} y_i X. \tag{3.3}$$

The string of values corresponding to the GCS after p iterations can now be written as

$$X_p = \begin{cases} (\mu_1, \mu_2, \mu_1) & \text{if } p=1 \\ X_1 X_{N-1} & \text{if } p>1. \end{cases} \tag{3.4}$$

It is now easy to prove that $C_{mn}(r)=$ constant for a set of values as in (3.1). For $j=1$ this implies proving that

$$\sum_{i=1}^{3^p} \mu^m(x_i)\mu^n(x_i+r) = \text{constant} \qquad \text{for } r=1,3,3^2,\ldots. \tag{3.5}$$

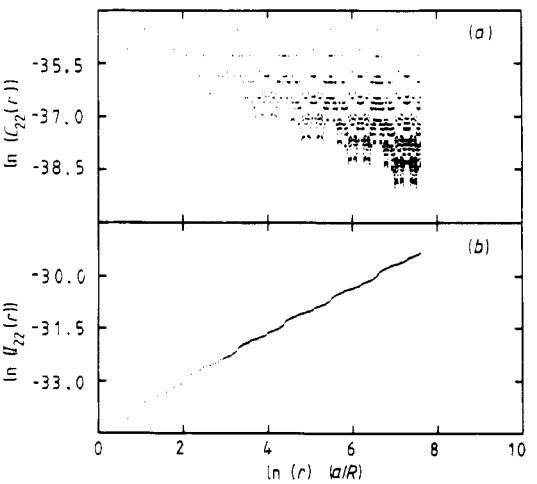

Figure 2. (a) Numerical calculation of $C_{mn}(r)$ for $m=n=2$ for the GCS as given by the definition (2.1). (b) Integral of $C_{mn}(r)$ as given by (4.1).

This is equivalent to showing that the corresponding string has the following permutation symmetry:

$$X_p = X_{p-p'}X_{p'} = X_{p'}X_{p-p'} \qquad \text{for } p' = 1, 2, \ldots . \tag{3.6}$$

This last property can be easily proved by induction from (3.4). This property of constancy of $C_{mn}(r)$ for some set of points does not, however, provide information on the average scaling behaviour because these constant values vary for different sets of points. The average scaling behaviour is quite clear in the integral of $C_{mn}(r)$ reported in figure 2(b) and it will be the subject of the next section.

4. Proof of the scaling behaviour of $C_{mn}(r)$ for the GCS

We discuss here the validity of the two assumptions that were made in § 2 in order to derive the scaling behaviour of (2.10). The first assumption corresponds to replacing the average of the products of terms corresponding to two different boxes with the product of the separate averages (see (2.3) and (2.4)). If one computes directly (2.4) for the GCS it can be seen (figures 2(a) and 3(a)) that this equation reduces drastically the fluctuations but it has exactly the same slope as that arising from the original definition of $C_{mn}(r)$ (2.1). If, however, we consider the integral of $C_{mn}(r)$ that has much less fluctuation (figures 2(b) and 3(b)), one can actually detect a very small shift in the amplitude of the order of 0.5%.

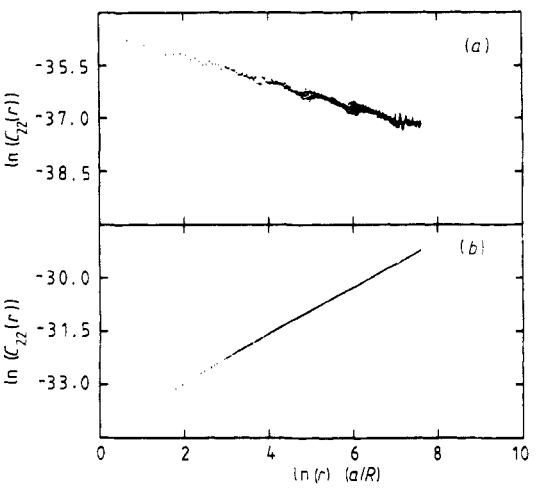

Figure 3. Same as figure 2 but $C_{mn}(r)$ is computed with the expressions corresponding to assumption I: (a) as given by equation (2.4); (b) as given by equation (4.3).

In order to understand these features it is convenient to study first the role of this assumption on the total number of pair products as functions of length scale in the integral of $C_{mn}(r)$. This integral $I_{mn}(r)$ can be written in a discrete form as

$$I_{mn}(r) = \sum_{r'=1}^{r} \langle\langle [\mu_{r'}(x_i^k)]^m [\mu_{r'}(x_i^{k+1})]^n \rangle\rangle = \frac{1}{N} \sum_{i=1}^{N} \mu(x_i)^m \sum_{j=1}^{r} \mu(x_{i+j})^n. \tag{4.1}$$

We now consider the question of how many pair products are present within a length scale $j = l$. It is clear from (4.1) that there will be N pairs for $l \leqslant r$ and zero pairs for

$l > r$. If we define the function $W(l)$ as the total number of pairs divided by the normalisation factor (in this case N), then we have

$$W(l) = \begin{cases} 1 & \text{for } l \leq r \\ 0 & \text{for } l > r. \end{cases} \tag{4.2}$$

We now consider the same problem for the integral of $C_{mn}(r)$ in the case when we make use of assumption I. We have

$$I_{mn}(r) = \sum_{r'=1}^{r} \langle\langle [\mu_{r'}(x_i^k)]^m \rangle \langle \mu_{r'}(x_i^{k+1})]^n \rangle\rangle$$

$$= \sum_{j=1}^{r} \frac{1}{Nj} \sum_{k=0}^{(N/j)-1} \left(\sum_{i=1}^{r} \mu(x_{i+k\cdot j})^m \right) \left(\sum_{i=1}^{r} \mu(x_{i+(k+1)j})^n \right). \tag{4.3}$$

The calculation of the function $W(l)$ is now complicated by the fact that the normalisation factor Nj depends explicitly on j. For l even we obtain

$$W(l) = \sum_{j=l/2+1}^{r} \frac{\min(2j-l, l)}{j^2} \tag{4.4}$$

while for l odd the minimum value of j is $\frac{1}{2}(l+1)$. For $l, r \gg 1$ we can take the continuum limit. This gives

$$W(l) = 2 \ln 2 - l/r \qquad \text{for } l < r \tag{4.5}$$

$$W(l) = 2 \ln(2r/l) + (l/r) - 2 \qquad \text{for } r < l < 2r \tag{4.6}$$

$$W(l) = 0 \qquad \text{for } l > 2r. \tag{4.7}$$

It is clear therefore that the shape of the functions given by (4.2) and (4.5)-(4.7) is different because the first one is just a step function while the second shows a continuous decay. The two functions, however, decay over the same characteristic length. For this reason the scaling properties are identical and only a small shift in the amplitude is produced by assumption I.

Assumption II consisted in the replacement of the index $k+1$ by k in the double chevron of (2.7). We now consider an explicit calculation of the function

$$F(r) = \langle\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \rangle\rangle \tag{4.8}$$

for different values of the box size r. Looking at figure 4 we can see that for $r = R/3$

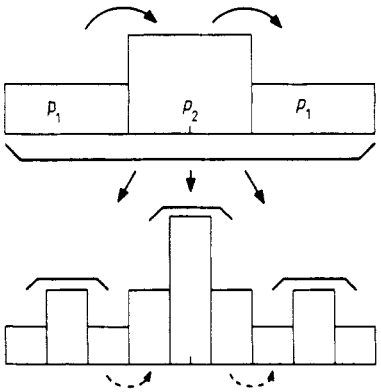

Figure 4. Schematic picture of the multiplication processes necessary to explain the scaling relation (4.12).

we have the largest non-trivial level of coarse graining of our distribution and the evaluation of (4.8) consists simply of the two products corresponding to the two curved full arrows in figure 4. This gives

$$F(\tfrac{1}{3}R) = \tfrac{1}{3}(p_1^m p_2^n + p_2^m p_1^n). \tag{4.9}$$

At the next level of coarse graining the internal arrows of the three blocks (figure 4) can be expressed in terms of $F(\frac{1}{3}R)$ and a similar relation can also be found for the two curved broken arrows that connect the three blocks. We can then write

$$F((\tfrac{1}{3})^2 R) = [\tfrac{1}{3}(2p_1^{m+n} + p_2^{m+n})F(\tfrac{1}{3}R) + \tfrac{1}{3}p_1^{m+n}F(\tfrac{1}{3}R)]. \tag{4.10}$$

By similar arguments it can be shown that the next level of coarse graining ($r = (\frac{1}{3})^3 R$) gives rise to

$$F((\tfrac{1}{3})^3 R) = [\tfrac{1}{3}(2p_1^{m+n} + p_2^{m+n})F((\tfrac{1}{3})^2 R) + (\tfrac{1}{3}p_1^{m+n})^2 F(\tfrac{1}{3}R)]. \tag{4.11}$$

These relations can be generalised for the pth level of coarse graining by the following iterative relation:

$$F((\tfrac{1}{3})^p R) = bF((\tfrac{1}{3})^{(p-1)}R) + c^{(p-1)}F(\tfrac{1}{3}R)$$
$$b = \tfrac{1}{3}(2p_1^{m+n} + p_2^{m+n}) \qquad c = \tfrac{1}{3}p_1^{m+n}. \tag{4.12}$$

This relation is equivalent to the more transparent scaling relation

$$F(\tfrac{1}{3}r) = (b+c)F(r) - bcF(3r). \tag{4.13}$$

If we now write down the corresponding transfer matrix it is easy to observe that b and c are the eigenvalues. Since $b > c$ the scaling relation (4.13) can be further simplified asymptotically into

$$F(\tfrac{1}{3}r) = bF(r). \tag{4.14}$$

The validity of (4.14) can be easily checked by reinserting it in (4.13).

The final scaling behaviour of $F(r)$ is therefore

$$F(r) = \langle\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \rangle\rangle \sim r^{(\ln b)/\ln(1/3)} = r^{\tau(m+n)+1} \tag{4.15}$$

where τ was introduced in (2.8). Clearly $\langle\langle \tilde{\mu}(r_k)^{m+n} \rangle\rangle$ has the same scaling as $F(r)$ which completes the proof of assumption II as far as scaling is concerned.

In order to study the effect of this assumption on the amplitude we consider the ratio

$$Q = \frac{\langle\langle \tilde{\mu}(r_k)^{m+n} \rangle\rangle}{\langle\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \rangle\rangle}. \tag{4.16}$$

For a generic value $r_k = (\frac{1}{3})^N R$ the numerator gives

$$\langle\langle \tilde{\mu}(r_k)^{(m+n)} \rangle\rangle = (\tfrac{1}{3}b)^N. \tag{4.17}$$

For the denominator one can use the scaling relation given by (4.13) (non-asymptotic). From the corresponding transfer matrix one obtains

$$\langle\langle \tilde{\mu}(r_k)^m \tilde{\mu}(r_{k+1})^n \rangle\rangle = \frac{b^N - c^N}{(b-c)}(p_1^m p_2^n + p_2^m p_1^n)3^{-N}. \tag{4.18}$$

In the limit $N \to \infty$ (note that $b > c$) we obtain finally

$$Q = \frac{p_1^{(m+n)} + p_2^{(m+n)}}{p_1^m p_2^n + p_2^m p_1^n} \qquad (>1). \tag{4.19}$$

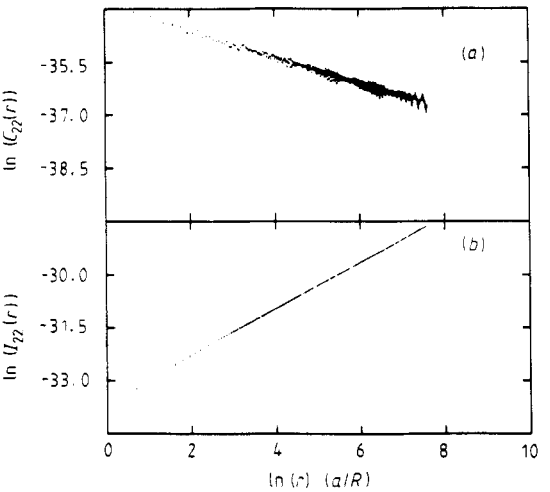

Figure 5. Numerical calculation of (*a*) $C_{mn}(r)$ and (*b*) $I_{mn}(r)$ after assumption II (equation (2.7)). Note that $I_{mn}(r)$ is shifted with respect to figure 2(*b*) just by the factor Q given by equation (4.19).

This shows therefore that assumption II leaves the scaling invariant but changes the amplitude by a non-trivial prefactor, explicitly dependent on m and n. In figure 5 we report a numerical calculation corresponding to assumption II that shows a shift in prefactor, with respect to the original correlation function, exactly as predicted by (4.19).

5. Discussion and summary

In conclusion we have analysed in detail the implicit assumptions that lie at the basis of the proposed scaling relation for multifractal correlations (Cates and Deutsch 1987). Our results show that the scaling behaviour of Cates and Deutsch (1987) is indeed correct. However, non-trivial properties of the amplitude can also be understood along the lines discussed here.

In our discussion we have mainly referred to the generalised Cantor set in one dimension. However, the results can be easily generalised for distributions embedded in d dimensions which have a generator. These are distributions for which there always exists a length scale r such that:

$$\chi(q, r)^2 \equiv \left(\sum_k \tilde{\mu}(r_k)^q \right)^2 = \chi(q, r^2). \tag{5.1}$$

One then readily obtains the general final result:

$$C_{mn}(r) \sim (R/a)^y (r/a)^z \tag{5.2}$$

$$y = -\tau(m+n) - d \tag{5.3}$$

$$z = \tau(m+n) - \tau(m) - \tau(n) - d. \tag{5.4}$$

One of us (Siebesma 1988) recently investigated the validity of (5.2)-(5.4) for self-similar distributions which do not possess the property (5.1) like the critical wavefunctions corresponding to an incommensurate potential (Siebesma and Pietronero 1987).

It is interesting to note that similar relations have also been derived for the scaling properties of correlations in Hamiltonian systems (Wegner 1985).

Finally it should be mentioned that the correlations discussed have a direct application in the field of fully developed turbulence in three dimensions. It provides a direct proof of the conjecture (Frisch *et al* 1987) which relates the energy dissipation correlation function with the structure functions. These are defined as:

$$\langle |\delta v_l(\boldsymbol{x})|^p \rangle \sim l^{\xi_p} \tag{5.5}$$

where $\delta v_l(\boldsymbol{x})$ denotes the velocity difference on a distance l at a point $\boldsymbol{x}$ and the chevrons imply an unnormalised spatial average. We want to relate (5.4) with the energy dissipation correlation function:

$$\langle \varepsilon_l(\boldsymbol{x}) \varepsilon_l(\boldsymbol{x}+\boldsymbol{r}) \rangle \sim r^{-\mu} \tag{5.6}$$

where $\varepsilon_l(\boldsymbol{x})$ is the rate of energy dissipation on an area of linear size l centred around $\boldsymbol{x}$. If we use the well established relation

$$\varepsilon_l(\boldsymbol{x}) \sim \delta v_l(x)^3 / l \tag{5.7}$$

then the application of the scaling results (5.1)-(5.3) directly gives (for $l \ll r$) the result

$$\mu = 2\xi_3 - \xi_6. \tag{5.8}$$

It can be easily checked that (5.7) holds for both the random β model (Benzi *et al* 1984) and the log-normal model (Kolmogorov 1962).

Acknowledgments

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